

LEFT AND RIGHT CENTERS IN QUASI-POISSON GEOMETRY OF MODULI SPACES

PAVOL ŠEVERA

ABSTRACT. We introduce left central and right central functions and left and right leaves in quasi-Poisson geometry, generalizing central (or Casimir) functions and symplectic leaves from Poisson geometry. They lead to a new type of (quasi-)Poisson reduction, which is both simpler and more general than known quasi-Hamiltonian reductions. We study these notions in detail for moduli spaces of flat connections on surfaces, where the quasi-Poisson structure is given by an intersection pairing on homology.

1. INTRODUCTION

A function on a Poisson manifold is *central* (or *Casimir*) if it Poisson-commutes with every function. A function is central iff it is constant on each *symplectic leaf* of the Poisson manifold.

Among the most interesting Poisson spaces are moduli spaces of flat connections on an oriented compact surface, with the Poisson structure of Atiyah-Bott [AB] and Goldman [G] given by an intersection pairing on the surface. Their symplectic leaves are obtained by fixing the conjugacy classes of the holonomies along the boundary circles.

Moduli spaces on surfaces with marked points on the boundary carry a *quasi-Poisson structure* [AMM, AKM]. A quasi-Poisson manifold is by definition a manifold with an action of a Lie algebra \mathfrak{g} and with an invariant bivector field π , satisfying

$$[\pi, \pi]/2 = \rho(\phi),$$

where $\rho(\phi)$ is a 3-vector field coming from an invariant inner product on \mathfrak{g} and from the structure constants of \mathfrak{g} . These moduli spaces have the advantage of being smooth and can be built out of simple pieces using the operation of fusion. Moduli spaces without marked points can then be obtained via a quasi-Hamiltonian reduction.

In the same way as Poisson structures are analogous to non-commutative algebras, quasi-Poisson structures are analogous to non-commutative algebras in a braided monoidal category. Motivated by this analogy, we introduce *left central* and *right central* functions, and the corresponding *left* and *right leaves* of quasi-Poisson manifolds. Besides the interesting geometry of these foliations, these notions bring a new type of reduction, the *central reduction* of quasi-Poisson manifolds, which produces symplectic and Poisson manifolds.

Despite the simplicity of central reduction (it is, in a way, simpler than the standard moment map reduction of symplectic manifolds), it contains as a special case the so far most general quasi-Hamiltonian reduction of [LS4]. Replacing moment maps with left/right centers also allows us to define the fusion of D/H -valued moment maps of [AK], which was so far missing.

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As we mentioned above, the motivating and also the most important examples of quasi-Poisson manifolds are moduli spaces of flat \mathfrak{g} -connections on a surface with marked points on the boundary. We reformulate these quasi-Poisson structures as an intersection pairing of homology with coefficients in a local system. This formulation is manifestly natural (it doesn't use any splitting of the surface to simpler pieces) and also very similar the formulation of Atiyah-Bott and Goldman. It also reduces all non-degeneracy problems to Poincaré duality, and left/right centers are found simply as holonomies along the parts of the boundary that don't intersect any cycle. Central reduction then produces interesting examples of symplectic and Poisson manifolds.

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2. LEAVES AND CENTRAL MAPS IN QUASI-POISSON GEOMETRY

Let \mathfrak{g} be a Lie algebra with an invariant element $t \in (S^2 \mathfrak{g})^{\mathfrak{g}}$. Let $\phi \in \bigwedge^3 \mathfrak{g}$ be given by $\phi = -[t^{1,2}, t^{2,3}]/4$, i.e.

$$\phi(\alpha, \beta, \gamma) = \frac{1}{4} \langle [t^\sharp \alpha, t^\sharp \beta], \gamma \rangle \text{ for all } \alpha, \beta, \gamma \in \mathfrak{g}^*.$$

The following definition is due to Alekseev, Kosmann-Schwarzbach, and Meinrenken [AKM].

Definition 1. A \mathfrak{g} -quasi-Poisson manifold is a manifold M with an action ρ of \mathfrak{g} and with a \mathfrak{g} -invariant bivector field π , satisfying

$$[\pi, \pi]/2 = \rho^{\otimes 3}(\phi).$$

A map $F : M \rightarrow M'$ between two \mathfrak{g} -quasi-Poisson manifolds is quasi-Poisson if it is \mathfrak{g} -equivariant and if $F_* \pi_M = \pi_{M'}$.

If (M, ρ, π) is a quasi-Poisson manifold, let

$$\sigma = \pi + \frac{1}{2} \rho^{\otimes 2}(t) \in \Gamma(T^{\otimes 2} M).$$

Definition 2. A function $f \in C^\infty(M)$ is left-central if $\sigma(df, \cdot) = 0$ and right-central if $\sigma(\cdot, df) = 0$.

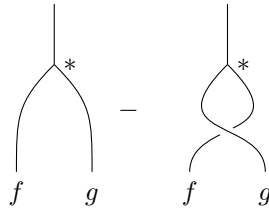
Remark. This definition, as well as many other things in this paper, is motivated by the following quantum analogue. Let Φ be a Drinfeld associator and let $U\mathfrak{g}\text{-mod}^\Phi$ be the category of $U\mathfrak{g}$ -modules with the braiding and the associativity constraint deformed by t and Φ (see [D1] for details). Suppose that \mathfrak{g} acts on M , and that $*$ is a star product on M making $C^\infty(M)$ to an associative algebra in $U\mathfrak{g}\text{-mod}^\Phi$, i.e.

$$* \circ (* \otimes 1) = * \circ (1 \otimes *) \circ (\Phi \cdot).$$

Then, as observed in [EE], there is a \mathfrak{g} -quasi-Poisson structure on M given by

$$\pi(df, dg) = (f * g - g * f)/\hbar \mod \hbar.$$

The \hbar -term of the braided commutator



is then equal to $\sigma(df, dg)$; the braiding contributes the symmetric part $\frac{1}{2} \rho^{\otimes 2}(t)$ of σ .

As a general rule, we shall only use those parts of quasi-Poisson geometry which have a clear quantum analogues, i.e. which make sense for algebras in the braided monoidal category $U\mathfrak{g}\text{-mod}^{\Phi}$.

Let us also remark here that one can define quasi-Poisson algebras in an arbitrary infinitesimally braided category \mathcal{C} , i.e. a linear symmetric monoidal category with a natural transformation $t^{X,Y} : X \otimes Y \rightarrow X \otimes Y$ satisfying the relations $t^{X,Y} = t^{Y,X}$ and $t^{X \otimes Y, Z} = t^{X,Z} + t^{Y,Z}$: a quasi-Poisson algebra A in \mathcal{C} is by definition a commutative algebra together with a skew-symmetric morphism $\{, \} : A \otimes A \rightarrow A$ satisfying the Leibniz rule and the quasi-Jacobi identity

$$\{\{.,.\},.\} + c.p. = -\frac{1}{4}[t^{A,A} \otimes \text{id}, \text{id} \otimes t^{A,A}] : A \otimes A \otimes A \rightarrow A.$$

Definition 3. A \mathfrak{g} -quasi-Poisson manifold is quasi-Poisson-commutative if $\sigma = 0$. Equivalently, it is a \mathfrak{g} -manifold such that the stabilizers of points are coisotropic Lie subalgebras of \mathfrak{g} , together with the bivector field $\pi = 0$.

Here a Lie subalgebra $\mathfrak{c} \subset \mathfrak{g}$ is called *coisotropic* if the image of t in $S^2(\mathfrak{g}/\mathfrak{c})$ vanishes. The most straightforward example of a quasi-Poisson-commutative manifold is G/C , where G is a Lie group with the Lie algebra \mathfrak{g} and $C \subset G$ is a closed subgroup with a coisotropic Lie algebra $\mathfrak{c} \subset \mathfrak{g}$.

Definition 4. If M is \mathfrak{g} -quasi-Poisson and N is \mathfrak{g} -quasi-Poisson-commutative, a map $F : M \rightarrow N$ is left (or right) central, if F is \mathfrak{g} -equivariant and if $F^*f \in C^\infty(M)$ is left (or right) central for every $f \in C^\infty(N)$.

Notice that a left (or right) central map is automatically quasi-Poisson.

We can characterize left (or right) central maps as follows. There are three natural integrable distributions on M :

$$T^L M = \text{the image of } a_L : T^*M \rightarrow TM, \alpha \mapsto \sigma(\cdot, \alpha),$$

$$T^R M = \text{the image of } a_R : T^*M \rightarrow TM, \alpha \mapsto \sigma(\alpha, \cdot),$$

$$T^{big} M = \text{the image of } a : \mathfrak{g} \oplus T^*M \rightarrow TM, (u, \alpha) \mapsto \rho(u) + \pi(\alpha, \cdot).$$

Their integrability follows from the fact that a_L , a_R , and a are the anchor maps for certain Lie algebroid structures (see [LS3] for a_L and a_R and [BC1] for a). Notice that

$$T^L M + \rho(\mathfrak{g}) = T^R M + \rho(\mathfrak{g}) = T^{big} M$$

and that σ gives a non-degenerate pairing

$$\sigma^{-1} : T^L M \times T^R M \rightarrow \mathbb{R}$$

given by

$$\sigma^{-1}(u, v) = \sigma(\alpha, \beta), \quad \alpha, \beta \in T^*M \text{ such that } v = \sigma(\alpha, \cdot), u = \sigma(\cdot, \beta).$$

This pairing is a generalization of the symplectic form on the symplectic leaves of a Poisson manifold: if $\mathfrak{g} = 0$ then $\sigma = \pi$ and $T^L M = T^R M$ is the tangent space of the symplectic leaves of the Poisson structure π , and σ^{-1} is the symplectic form on $T^L M = T^R M$.

Definition 5. The integral leaves of $T^L M$ are the left leaves of M , the integral leaves of $T^R M$ are the right leaves of M , and the integral leaves of $T^{big} M$ are the big leaves of M .

Notice that a function is left-central (right-central) iff it is constant on the left (right) leaves. An equivariant map $F : M \rightarrow N$ is thus left (or right) central iff it is constant on each left (or right) leaf.

Since σ is \mathfrak{g} -invariant, each of the three foliations is \mathfrak{g} -invariant. Big leaves are (minimal) \mathfrak{g} -quasi-Poisson submanifolds of M . Left and right leaves are contained within the big leaves; they are (in general) not quasi-Poisson submanifolds (see, however, Theorem 10). Let us also notice that while foliation by big leaves can be singular (the dimension of big leaves can jump), the foliation of a given big leaf by left (or right) leaves is a true foliation:

Proposition 1. *The action of \mathfrak{g} on the space of left (or right) leaves contained within a single big leaf $Y \subset M$ is locally transitive; in particular, left leaves form a foliation of Y . The stabilizer of any left (or right) leaf is a coisotropic subalgebra.*

Proof. Transitivity follows from $T^L M + \rho(\mathfrak{g}) = T^{big} M$. The local space of left (or right) leaves within Y is quasi-Poisson-commutative, so the stabilizers must be coisotropic. \square

Let us now single out quasi-Poisson manifolds with the most interesting left and right centers.

Definition 6. *A \mathfrak{g} -quasi-Poisson manifold M is split-symplectic if $t \in S^2 \mathfrak{g}$ is non-degenerate, if $T^{big} M = TM$ (i.e. M has a single big leaf), and if the stabilizers of the left (or equivalently right) leaves are Lagrangian.*

Here we call a Lie subalgebra $\mathfrak{c} \subset \mathfrak{g}$ *Lagrangian* (provided t is non-degenerate, so that it defines a pairing on \mathfrak{g}) if $\mathfrak{c}^\perp = \mathfrak{c}$. It implies that $\dim \mathfrak{g}$ is even, as $\dim \mathfrak{g} = 2 \dim \mathfrak{c}$. The split-symplectic condition says that the dimension of the left (and thus also of the right) leaves is as small as possible (supposing $T^{big} M = TM$ and non-degenerate t), namely

$$\text{rank } T^L M = \text{rank } T^R M = \dim M - \frac{1}{2} \dim \mathfrak{g}.$$

Example 1. Suppose that $t_{\mathfrak{g}} \in (S^2 \mathfrak{g})^{\mathfrak{g}}$ is non-degenerate. Let $\bar{\mathfrak{g}}$ be \mathfrak{g} with $t_{\bar{\mathfrak{g}}} = -t_{\mathfrak{g}}$, and let $\mathfrak{d} = \mathfrak{g} \oplus \bar{\mathfrak{g}}$, with $t_{\mathfrak{d}} = t_{\mathfrak{g}} \oplus t_{\bar{\mathfrak{g}}}$.

If G is a connected group integrating \mathfrak{g} , then G with the action of \mathfrak{d} , $(u, v) \mapsto u^L - v^R$, is a \mathfrak{d} -quasi-Poisson-commutative manifold (here u^L and v^R are the left and right invariant vector fields on G corresponding to $u, v \in \mathfrak{g}$).

Let us now consider $G \times G$ with the diagonal action of \mathfrak{d} . The bivector field

$$\pi = \sum_{ij} t^{ij} ((0, e_i^L) \wedge (e_j^L, 0) - (0, e_i^R) \wedge (e_j^R, 0))$$

(e_i is a basis of \mathfrak{g} and $t_{\mathfrak{g}} = \sum_{ij} t^{ij} e_i \otimes e_j$) makes it to a \mathfrak{d} -quasi-Poisson manifold. As

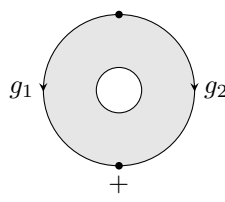
$$\sigma = \sum_{ij} t^{ij} ((0, e_i^L) \otimes (e_j^L, 0) - (0, e_i^R) \otimes (e_j^R, 0)),$$

the projection $p_1 : G \times G \rightarrow G$ to the first factor is left-central and the projection p_2 to the second factor is right-central.

The big leaves in $G \times G$ are $m^{-1}(\mathcal{O})$, where $m : G \times G \rightarrow G$, $(g_1, g_2) \mapsto g_1 g_2^{-1}$, and $\mathcal{O} \subset G$ runs over conjugacy classes of G . For every conjugacy class $\mathcal{O} \subset G$ the manifold $m^{-1}(\mathcal{O})$ is split-symplectic, and its left (resp. right) leaves are the fibres of p_1 (resp. p_2).

The \mathfrak{d} -quasi-Poisson manifold $G \times G$ can be described as the fusion product $G \circledast G$ (see Section 4). As we shall see in Section 5, it can also be interpreted as

the moduli space of flat G -bundles on the annulus with two marked points on the exterior circle:



The submanifold $m^{-1}(\mathcal{O})$ is given by restricting the holonomy along the internal circle to be in \mathcal{O} .

3. CENTRAL REDUCTION OF QUASI-POISSON MANIFOLDS

If $\mathfrak{c} \subset \mathfrak{g}$ is a coisotropic Lie subalgebra, i.e. if the image of $t \in S^2\mathfrak{g}$ in $S^2(\mathfrak{g}/\mathfrak{c})$ vanishes, then also the image of $\phi \in \bigwedge^3 \mathfrak{g}$ in $\bigwedge^3(\mathfrak{g}/\mathfrak{c})$ vanishes. As a consequence, we get the following result.

Proposition 2 ([LS1]). *If M is a \mathfrak{g} -quasi-Poisson manifold then the algebra of \mathfrak{c} -invariant functions $C^\infty(M)^\mathfrak{c} \subset C^\infty(M)$ is a Poisson algebra, with the Poisson bracket*

$$\{f, g\} = \pi(df, dg) = \sigma(df, dg)$$

inherited from $C^\infty(M)$. In particular, if M/\mathfrak{c} is a manifold then it is Poisson.

Remark. By M/\mathfrak{c} being a manifold we mean the following: there is a manifold M' and a surjective submersion $M \rightarrow M'$ such that its fibres are the \mathfrak{c} -orbits on M .

The following observation is obvious, but also central for this section.

Proposition 3. *Let M be a \mathfrak{g} -quasi-Poisson manifold and $f \in C^\infty(M)$ a left (or right) central function. If f is \mathfrak{c} -invariant for some coisotropic $\mathfrak{c} \subset \mathfrak{g}$ then it is central (i.e. Casimir) in the Poisson algebra $C^\infty(M)^\mathfrak{c}$.*

Proof. If $g \in C^\infty(M)^\mathfrak{c}$ then $\{f, g\} = \sigma(df, dg) = 0$, as f is left central. The proof for right-central f 's is similar. \square

Let us call a diagram

$$\begin{array}{ccc} & M & \\ \mu_L \swarrow & & \searrow \mu_R \\ N_L & & N_R \end{array}$$

a *central pair* if M is a \mathfrak{g} -quasi-Poisson manifold and μ_L and μ_R are a left and a right central map respectively.

As a version of Proposition 3 we get the following reduction method.

Theorem 1 (Central reduction). *Let*

$$(1) \quad \begin{array}{ccc} & M & \\ \mu_L \swarrow & & \searrow \mu_R \\ N_L & & N_R \end{array}$$

be a central pair. Let $\mathfrak{c} \subset \mathfrak{g}$ be a coisotropic Lie subalgebra and let $\mathcal{O}_L \subset N_L$ and $\mathcal{O}_R \subset N_R$ be \mathfrak{c} -invariant submanifolds. If $\mu_L \times \mu_R$ is transverse to $\mathcal{O}_L \times \mathcal{O}_R$ and if M/\mathfrak{c} is a manifold then

$$(2) \quad (\mu_L^{-1}(\mathcal{O}_L) \cap \mu_R^{-1}(\mathcal{O}_R))/\mathfrak{c} \subset M/\mathfrak{c}$$

is a Poisson submanifold.

Proof. The Poisson bivector field π_{red} on M/\mathfrak{c} is given by

$$\pi_{red}(\alpha, \beta) = \pi(\alpha, \beta) = \sigma(\alpha, \beta)$$

for any $\alpha, \beta \in T_x^*M$ which descend to $T_{[x]}^*(M/\mathfrak{c})$, i.e. which are in the kernel of $i_{\rho(u)}$ for every $u \in \mathfrak{c}$. If α is the pull-back of a covector from N_L then by left-centrality of μ_L we have $\sigma(\alpha, \cdot) = 0$. This shows that $\pi_{red}(\alpha, \cdot) = 0$ whenever α is the pull-back of an element of the conormal bundle of \mathcal{O}_L . A similar argument applies to \mathcal{O}_R , hence (2) is indeed a Poisson submanifold. \square

A natural question is how to describe the symplectic leaves of the Poisson manifold M/\mathfrak{c} . Theorem 1 does it under the following circumstances.

Definition 7. A central pair (1) is split-symplectic if $t \in S^2\mathfrak{g}$ is non-degenerate, $T^{big}M = TM$ (i.e. M has just one big leaf), and the actions of \mathfrak{g} on N_L and N_R are transitive with Lagrangian stabilizers.

Remark. If (1) is a split-symplectic central pair then M is a split-symplectic \mathfrak{g} -quasi-Poisson manifold, and the fibers of μ_L (μ_R) are the left (right) leaves of M . If, on the other hand, M is a split-symplectic \mathfrak{g} -quasi-Poisson manifold then we can set N_L (N_R) to be the local space of left (right) leaves of M , and get (locally) a split-symplectic central pair. Split-symplectic central pairs and split-symplectic quasi-Poisson manifolds are thus essentially the same thing.

Theorem 2 (Split-symplectic central reduction). *Suppose, in the context of Theorem 1, that (1) is split-symplectic, $\mathfrak{c} \subset \mathfrak{g}$ is Lagrangian, and \mathcal{O}_L and \mathcal{O}_R are \mathfrak{c} -orbits. Then*

$$(\mu_L^{-1}(\mathcal{O}_L) \cap \mu_R^{-1}(\mathcal{O}_R))/\mathfrak{c} \subset M/\mathfrak{c}$$

is a symplectic leaf.

Proof. It follows from Proposition 6 in Appendix, where we set $V = T_x^*M$, $W = \mathfrak{g}^*$, $f = \rho^*$, and $C = \text{Ann } \mathfrak{c}$. \square

Example 2. Let $\mathcal{O} \subset G$ be a conjugacy class. Let us consider the split-symplectic central pair

$$\begin{array}{ccc} & m^{-1}(\mathcal{O}) & \\ p_1 \swarrow & & \searrow p_2 \\ G & & G \end{array}$$

defined in Example 1. Notice that $\mathfrak{g}_{diag} \subset \mathfrak{g} \oplus \bar{\mathfrak{g}}$ is a Lagrangian Lie subalgebra. Let us choose a \mathfrak{g}_{diag} -orbit, i.e. a conjugacy class, in each G ; we shall denote them \mathcal{O}_L and \mathcal{O}_R as above. Central reduction gives us a symplectic form on (the non-singular part of)

$$\{(g_1, g_2, g_3) \in \mathcal{O}_L \times \mathcal{O}_R \times \mathcal{O}; g_1 = g_3 g_2\}/G$$

where G acts by conjugation. This symplectic space can be identified with the moduli space of flat G -bundles over a sphere with 3 punctures, with the holonomies around the punctures in \mathcal{O}_L , \mathcal{O}_R , \mathcal{O} .

Partial reduction. The reduction by coisotropic Lie algebra we described above is a special case of a reduction of quasi-Poisson manifolds to quasi-Poisson manifolds.

For any Lie subalgebra $\mathfrak{c} \subset \mathfrak{g}$ let

$$\mathfrak{c}^\perp = t(\text{Ann } \mathfrak{c}, \cdot) \subset \mathfrak{g}.$$

The Lie subalgebra \mathfrak{c} is coisotropic iff $\mathfrak{c}^\perp \subset \mathfrak{c}$; in that case $\mathfrak{c}^\perp \subset \mathfrak{c}$ is an ideal. The element $t \in S^2\mathfrak{g}$ descends to an element $t' \in S^2(\mathfrak{c}/\mathfrak{c}^\perp) \subset S^2(\mathfrak{g}/\mathfrak{c}^\perp)$ via the projection

$S^2\mathfrak{g} \rightarrow S^2(\mathfrak{g}/\mathfrak{c}^\perp)$. The element $\phi \in \bigwedge^3 \mathfrak{g}$ descend in the same way to the element $\phi' \in \bigwedge^3(\mathfrak{c}/\mathfrak{c}^\perp)$ corresponding to t' . The same applies if we replace $\mathfrak{c}/\mathfrak{c}^\perp$ with $\mathfrak{c}/\mathfrak{h}$ for any ideal $\mathfrak{h} \subset \mathfrak{c}$ containing \mathfrak{c}^\perp . As a result we get the following.

Proposition 4 ([LS1]). *If M is a \mathfrak{g} -quasi-Poisson manifold, $\mathfrak{c} \subset \mathfrak{g}$ a coisotropic Lie subalgebra, and $\mathfrak{h} \subset \mathfrak{c}$ an ideal such that $\mathfrak{c}^\perp \subset \mathfrak{h}$, and if M/\mathfrak{h} is a manifold, then M/\mathfrak{h} is $\mathfrak{c}/\mathfrak{h}$ -quasi-Poisson, with the bivector field pushed-forward from M .*

A useful particular case is when \mathfrak{g} is a direct sum $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ with $t = t_1 + t_2$, $t_i \in (S^2\mathfrak{g}_i)^{\mathfrak{g}_i}$, and $\mathfrak{c}_1 \subset \mathfrak{g}_1$ is a coisotropic Lie subalgebra. If we set $\mathfrak{h} = \mathfrak{c}_1$ and $\mathfrak{c} = \mathfrak{c}_1 \oplus \mathfrak{g}_2$ then Proposition 4 says that M/\mathfrak{c}_1 is \mathfrak{g}_2 -quasi-Poisson.

Theorems 1 and 2 have now the following extensions (we omit the proofs, as they are very similar).

Theorem 3 (Central reduction 2). *Suppose that (1) is a central pair. Let $\mathfrak{c} \subset \mathfrak{g}$ be a coisotropic Lie subalgebra and $\mathfrak{h} \subset \mathfrak{c}$ an ideal containing \mathfrak{c}^\perp . Let $\mathcal{O}_L \subset N_L$ and $\mathcal{O}_R \subset N_R$ be \mathfrak{c} -invariant submanifolds. If $\mu_L \times \mu_R$ is transverse to $\mathcal{O}_L \times \mathcal{O}_R$ and if M/\mathfrak{h} , $\mathcal{O}_L/\mathfrak{h}$ and $\mathcal{O}_R/\mathfrak{h}$ are manifolds then*

$$M' := (\mu_L^{-1}(\mathcal{O}_L) \cap \mu_R^{-1}(\mathcal{O}_R))/\mathfrak{h} \subset M/\mathfrak{h}$$

is a $\mathfrak{c}/\mathfrak{h}$ -quasi-Poisson submanifold, and

$$\begin{array}{ccc} & M' & \\ \swarrow & & \searrow \\ \mathcal{O}_L/\mathfrak{h} & & \mathcal{O}_R/\mathfrak{h} \end{array}$$

is a central pair.

Theorem 4 (Split-symplectic central reduction 2). *If, in the context of Theorem 3, the central pair (1) is split symplectic, if $\mathfrak{h} = \mathfrak{c}^\perp$, and if \mathcal{O}_L and \mathcal{O}_R are \mathfrak{c} -orbits, then*

$$\begin{array}{ccc} & M' & \\ \swarrow & & \searrow \\ \mathcal{O}_L/\mathfrak{c}^\perp & & \mathcal{O}_R/\mathfrak{c}^\perp \end{array}$$

is a split-symplectic central pair.

Example 3. Let us consider again the split-symplectic $\mathfrak{g} \oplus \bar{\mathfrak{g}}$ -central pair

$$\begin{array}{ccc} & m^{-1}(\mathcal{O}) & \\ p_1 \swarrow & & \searrow p_2 \\ G & & G \end{array}$$

defined in Example 1. Suppose that \mathfrak{g} is semisimple and that $\mathfrak{b} \subset \mathfrak{g}$ is a Borel subalgebra. We shall reduce this central pair by the coisotropic subalgebra $\mathfrak{c} := \mathfrak{b} \oplus \mathfrak{b} \subset \mathfrak{g} \oplus \bar{\mathfrak{g}}$. Notice that $\mathfrak{c}^\perp = \mathfrak{n} \oplus \mathfrak{n}$, where $\mathfrak{n} \subset \mathfrak{b}$ is the nilpotent radical, and that $\mathfrak{c}/\mathfrak{c}^\perp = \mathfrak{t} \oplus \bar{\mathfrak{t}}$, where $\mathfrak{t} \subset \mathfrak{g}$ is the Cartan subalgebra.

Let us observe that \mathfrak{c} -orbits in G give us the Bruhat decomposition of G : for any element w of the Weyl group W_G we have the orbit $\mathcal{O}_w = BwB \subset G$, where B is the Borel subgroup.

We can identify the quotient $\mathcal{O}_w/\mathfrak{c}^\perp$ with $Tw \subset G$, where T is the Cartan subgroup. As a result, for any pair $w_1, w_2 \in W_G$ Theorem 4 gives us a split-symplectic $\mathfrak{t} \oplus \bar{\mathfrak{t}}$ -central pair

$$\begin{array}{ccc} & M_{w_1, w_2} & \\ p_1 \swarrow & & \searrow p_2 \\ Tw_1 & & Tw_2 \end{array}$$

As $\mathfrak{t} \oplus \bar{\mathfrak{t}}$ is Abelian, the quasi-Poisson structure on M_{w_1, w_2} is actually Poisson.

4. FUSION

If M_1 and M_2 are \mathfrak{g} -quasi-Poisson manifolds then their product $M_1 \times M_2$ (with $\pi_1 + \pi_2$) is $\mathfrak{g} \oplus \mathfrak{g}$ -quasi-Poisson. To make it \mathfrak{g} -quasi-Poisson, where $\mathfrak{g} \hookrightarrow \mathfrak{g} \oplus \mathfrak{g}$ is the diagonal inclusion, one needs to use a twist (the reason is that $\mathfrak{g}_{\text{diag}} \subset \mathfrak{g} \oplus \mathfrak{g}$ is not a quasi-Lie sub-bialgebra, but becomes so after the twist). The result is as follows.

Definition 8 ([AKM]). *The fusion product $M_1 \circledast M_2$ of two \mathfrak{g} -quasi-Poisson manifolds (M_1, ρ_1, π_1) and (M_2, ρ_2, π_2) is the manifold $M_1 \times M_2$ with the \mathfrak{g} -quasi-Poisson structure given by the diagonal action of \mathfrak{g} and by the bivector field*

$$(3) \quad \pi_{\circledast} = \pi_1 + \pi_2 - \frac{1}{2}(\rho_1 \wedge \rho_2)(t).$$

Let us notice that the corresponding tensor field σ_{\circledast} on $M_1 \circledast M_2$ is given by

$$(4) \quad \sigma_{\circledast} = \sigma_1 + \sigma_2 + (\rho_2 \otimes \rho_1)(t).$$

The fusion product makes the category of \mathfrak{g} -quasi-Poisson manifolds to a (non-braided) monoidal category.

Remark. Monoids in a braided monoidal category form a (non-braided) monoidal category: if A and B are monoids then $A \otimes B$ is a monoid as well, with the product

$$(A \otimes B) \otimes (A \otimes B) \rightarrow A \otimes B$$

given by the diagram

$$\begin{array}{ccccc} & A & & B & \\ & | & & | & \\ & \swarrow & & \searrow & \\ & & \times & & \\ & \nwarrow & & \nearrow & \\ A & B & & A & B \end{array}$$

As observed in [LS2], the fusion product is the semi-classical limit of the tensor product of monoids in the braided monoidal category $U\mathfrak{g}\text{-mod}^\Phi$. A similar observation holds for quasi-Poisson algebras in any infinitesimally braided category.

Slightly more generally, suppose that \mathfrak{g} has an invariant element $t_{\mathfrak{g}} \in (S^2\mathfrak{g})^{\mathfrak{g}}$ and that \mathfrak{h} is another Lie algebra with an element $t_{\mathfrak{h}} \in (S^2\mathfrak{h})^{\mathfrak{h}}$. For the Lie algebra $\mathfrak{g} \oplus \mathfrak{g} \oplus \mathfrak{h}$ we use the element $t_{\mathfrak{g} \oplus \mathfrak{g} \oplus \mathfrak{h}} = t_{\mathfrak{g}} \oplus t_{\mathfrak{g}} \oplus t_{\mathfrak{h}}$ and for $\mathfrak{g} \oplus \mathfrak{h}$ we use $t_{\mathfrak{g} \oplus \mathfrak{h}} = t_{\mathfrak{g}} \oplus t_{\mathfrak{h}}$.

Definition 9 ([AKM]). *If (M, ρ, π) is a $\mathfrak{g} \oplus \mathfrak{g} \oplus \mathfrak{h}$ -quasi-Poisson manifold then the (internal) fusion of M is M with the $\mathfrak{g} \oplus \mathfrak{h}$ -quasi-Poisson structure given by*

$$\rho_{\circledast}(u, v) = \rho(u, u, v) \quad (\forall u \in \mathfrak{g}, v \in \mathfrak{h})$$

and

$$\pi_{\circledast} = \pi - \frac{1}{2}(\rho_1 \wedge \rho_2)(t_{\mathfrak{g}})$$

where ρ_1 and ρ_2 are the actions of the first and of the second \mathfrak{g} in $\mathfrak{g} \oplus \mathfrak{g} \oplus \mathfrak{h}$ respectively.

Again, σ_{\otimes} is given by

$$(5) \quad \sigma_{\otimes} = \sigma + (\rho_2 \otimes \rho_1)(t_{\mathfrak{g}}).$$

Fusion is compatible with central maps in the following way.

Theorem 5. *Let*

$$\begin{array}{ccc} & M_1 & \\ \mu_L \swarrow & & \searrow \mu_R \\ N & & N' \end{array} \quad \begin{array}{ccc} & M_2 & \\ \nu_L \swarrow & & \searrow \nu_R \\ N' & & N'' \end{array}$$

be central pairs of \mathfrak{g} -quasi-Poisson manifolds. If $\mu_R \times \nu_L$ is transverse to the diagonal $N'_{diag} \subset N' \times N'$ (in particular, if one of μ_R, ν_L is a submersion) then the fibre product

$$M_1 \times_{N'} M_2 \subset M_1 \times M_2$$

over μ_R and ν_L is a \mathfrak{g} -quasi-Poisson submanifold of $M_1 \otimes M_2$ and

$$\begin{array}{ccc} & M_1 \times_{N'} M_2 & \\ \mu_L \swarrow & & \searrow \nu_R \\ N & & N'' \end{array}$$

is a central pair.

Proof. The submanifold $M_1 \times_{N'} M_2 \subset M_1 \times M_2$ is \mathfrak{g} -invariant. To show that it is a quasi-Poisson submanifold, we need to check that

$$\sigma_{\otimes}(\mu_R^* \gamma - \nu_L^* \gamma, \cdot) = 0$$

for every $(x, y) \in M_1 \times_{N'} M_2 \subset M_1 \times M_2$ and for every $\gamma \in T_z^* N'$ where $z = \mu_R(x) = \nu_L(y)$. Centrality gives us

$$\sigma_2(\nu_L^* \gamma, \cdot) = 0$$

$$\sigma_1(\mu_R^* \gamma, \cdot) = \sigma_1(\mu_R^* \gamma, \cdot) + \sigma_1(\cdot, \mu_R^* \gamma) = (\rho_2^{\otimes 2}(t))(\cdot, \mu_R^* \gamma) = ((\rho_2 \otimes \rho_{N'})(t))(\gamma, \cdot).$$

Since

$$((\rho_2 \otimes \rho_1)(t))(\mu_R^* \gamma - \nu_L^* \gamma, \cdot) = ((\rho_2 \otimes \rho_1)(t))(-\nu_L^* \gamma, \cdot) = -((\rho_2 \otimes \rho_{N'})(t))(\gamma, \cdot),$$

the expression (4) for σ_{\otimes} gives us

$$\sigma_{\otimes}(\mu_R^* \gamma - \nu_L^* \gamma, \cdot) = 0,$$

as we needed.

To finish the proof that we have a central pair we need to check that μ_L and ν_R are left and right central (respectively) on $M_1 \otimes M_2$. For $\alpha \in \Omega^1(N')$ we get $\sigma_{\otimes}(\mu_L^* \alpha, \cdot) = (\sigma_1 + \sigma_2 + (\rho_2 \otimes \rho_1)(t))(\mu_L^* \alpha, \cdot) = 0$ (the σ_1 -term vanishes because $\mu_L : M_2 \rightarrow N$ is left-central, the σ_2 and $\rho_2 \otimes \rho_1$ terms are obviously zero), hence $\mu_L : M_1 \otimes M_2 \rightarrow N$ is indeed left-central. The proof of right centrality of ν_R is similar. \square

5. QUASI-POISSON STRUCTURE ON MODULI SPACES AND ITS CENTERS

5.1. The Poisson structure of Atiyah–Bott and Goldman. Let us first recall the Poisson structure of Atiyah–Bott [AB] and Goldman [G] on moduli spaces of flat connections on a surface. Let \mathfrak{g} be a Lie algebra with an invariant element $t \in (S^2\mathfrak{g})^{\mathfrak{g}}$, G a connected Lie group with the Lie algebra \mathfrak{g} , and Σ an oriented compact surface, possibly with a boundary.

Let $P \rightarrow \Sigma$ be a principal G -bundle with a flat connection A and \mathfrak{g}_P the associated adjoint vector bundle. The bundle \mathfrak{g}_P inherits the flat connection A , i.e. we can see it as a local system on Σ . We shall denote this flat vector bundle by $\mathfrak{g}_{P,A}$.

The element $t : \mathfrak{g}^* \times \mathfrak{g}^* \rightarrow \mathbb{R}$ gives us a pairing $\mathfrak{g}_{P,A}^* \times \mathfrak{g}_{P,A}^* \rightarrow \mathbb{R}$, which in turn gives an intersection pairing on homology

$$(6) \quad \pi : H_1(\Sigma; \mathfrak{g}_{P,A}^*) \times H_1(\Sigma; \mathfrak{g}_{P,A}^*) \rightarrow \mathbb{R}.$$

Let now

$$M_{\Sigma}(G) = \text{Hom}(\pi_1(\Sigma), G)/G$$

be the moduli space of flat connections on principal G -bundles over Σ . The (formal) tangent space of $M_{\Sigma}(G)$ at $[P, A]$ is

$$(7a) \quad T_{[P,A]}M_{\Sigma}(G) = H^1(\Sigma; \mathfrak{g}_{P,A}),$$

the cotangent space is thus

$$(7b) \quad T_{[P,A]}^*M_{\Sigma}(G) = H_1(\Sigma; \mathfrak{g}_{P,A}^*),$$

and the intersection pairing (6) becomes a bivector field on $M_{\Sigma}(G)$. It is the Poisson structure of Atiyah–Bott and Goldman.

When Σ is closed and t is non-degenerate then the pairing (6) is non-degenerate by Poincaré duality. The Poisson structure is symplectic in this case, the symplectic form ω is equal to the corresponding intersection pairing on $H^1(\Sigma; \mathfrak{g}_{P,A})$, and can thus be expressed in terms of 1-forms as

$$\omega([\alpha], [\beta]) = \int_{\Sigma} \langle \alpha \wedge \beta \rangle \quad ([\alpha], [\beta] \in H^1(\Sigma; \mathfrak{g}_{P,A})),$$

where \langle, \rangle is the pairing on \mathfrak{g} coming from t .

5.2. Quasi-Poisson structures on moduli spaces. The most important examples of quasi-Poisson manifolds are moduli spaces of flat connections on a surface with marked points on the boundary [AMM, AKM]. Here we present these quasi-Poisson structures in terms of intersection pairing, as in Section 5.1. This point of view is significantly simpler than that of *op. cit.* and reduces the problem of finding left and right leaves to Poincaré duality. Let us remark that, unlike *op. cit.*, we don't discuss moment maps, as we replace them with central maps and pairs (see Section 6 for the relations between moment maps and central pairs).

Let Σ be an oriented compact surface with boundary and let $V \subset \partial\Sigma$ be a finite set. For simplicity we suppose that V meets every component of Σ .

Let

$$M_{\Sigma,V}(G) = \text{Hom}(\Pi_1(\Sigma, V), G)$$

where $\Pi_1(\Sigma, V)$ is the fundamental groupoid of Σ with the base set V . It is the moduli space of flat connections on principal G -bundles over Σ , trivialized over V .

The set $M_{\Sigma,V}(G)$ is naturally a smooth manifold (it can be identified with G^E , where E is the edge set of a graph Γ embedded to Σ with vertex set V , such that the embedding $\Gamma \subset \Sigma$ is a homotopy equivalence), with a natural action of G^V , given by

$$(g \cdot f)(\gamma) = g_{\text{head}(\gamma)} f(\gamma) g_{\text{tail}(\gamma)}^{-1} \quad (f : \Pi_1(\Sigma, V) \rightarrow G, \gamma \in \Pi_1(\Sigma, V), g \in G^V),$$

or equivalently by changing the trivializations of the principal bundles over V .

Similar to (7b), the cotangent space of $M_{\Sigma,V}(G)$ is the relative homology

$$(8) \quad T_{[P,A]}^* M_{\Sigma,V}(G) = H_1(\Sigma, V; \mathfrak{g}_{P,A}^*).$$

We can define an “intersection pairing” on $H_1(\Sigma, V; \mathfrak{g}_{P,A}^*)$ in the following way. Let us split V to two disjoint subsets $V = V_+ \sqcup V_-$. Let us move every point in V_+ a little along $\partial\Sigma$ in the direction given by the orientation induced from Σ , and every point in V_- in the opposite direction. Let us denote the set of moved points by \hat{V} . Since V and \hat{V} are disjoint, we have a well-defined intersection pairing

$$H_1(\Sigma, V; \mathfrak{g}_{P,A}^*) \times H_1(\Sigma, \hat{V}; \mathfrak{g}_{P,A}^*) \rightarrow \mathbb{R}.$$

As \hat{V} was obtained from V by an isotopy, we have a natural isomorphism

$$H_1(\Sigma, V; \mathfrak{g}_{P,A}^*) \cong H_1(\Sigma, \hat{V}; \mathfrak{g}_{P,A}^*).$$

Composing it with the intersection pairing we get the pairing

$$\sigma_{V_+, V_-} : H_1(\Sigma, V; \mathfrak{g}_{P,A}^*) \times H_1(\Sigma, V; \mathfrak{g}_{P,A}^*) \rightarrow H_1(\Sigma, V; \mathfrak{g}_{P,A}^*) \times H_1(\Sigma, \hat{V}; \mathfrak{g}_{P,A}^*) \rightarrow \mathbb{R},$$

which can be viewed via (8) as a tensor field on $M_{\Sigma,V}(G)$. (Perhaps in simpler terms: the intersection pairing on $H_1(\Sigma, V; \mathfrak{g}_{P,A}^*)$ is not well defined, as it is not clear how to count the intersections at V . The pairing σ_{V_+, V_-} is defined by a particular rule saying which of these intersections are counted (those that survive the isotopy) and which are not (those that disappear).)

The skew-symmetric part of the pairing σ_{V_+, V_-}

$$(9) \quad \pi = \frac{1}{2}(\sigma_{V_+, V_-} - \sigma_{V_+, V_-}^{op})$$

is independent of the decomposition $V = V_+ \sqcup V_-$; on cycles it is given by the rule that intersections at the points of V are counted with weight $1/2$.

The following theorem was essentially proven in [MT] and [LS1], though using a somewhat different language of “homotopy intersection pairing”. Intersection pairing with the local system $\mathfrak{g}_{P,A}^*$ seems to be better suited for our needs. Let $\bar{\mathfrak{g}} = \mathfrak{g}$, with t replaced by $-t$.

Theorem 6. *The bivector field π given by (9) defines a $\mathfrak{g}^{V_+} \oplus \bar{\mathfrak{g}}^{V_-}$ -quasi-Poisson structure on $M_{\Sigma,V}(G)$. The tensor field σ_{V_+, V_-} is the corresponding σ -tensor.*

Proof. This proof is adapted from [LS1, Theorems 2 and 3].

The theorem is valid if $(\Sigma, V_+ \sqcup V_-)$ is a disjoint union of disks, with one point in V_+ and one point in V_- on each disk. In this case $\sigma_{V_+, V_-} = 0$ and $M_{\Sigma,V}(G)$ is a quasi-Poisson-commutative manifold.

If the theorem is valid for a surface $(\Sigma, V = V_+ \sqcup V_-)$ then it is also valid for $(\Sigma', V' = V'_+ \sqcup V'_-)$, for these (Σ', V') :

- (1) $\Sigma' = \Sigma$, $V' = V - \{x\}$ for some $x \in V$. In this case $M_{\Sigma', V'}(G) = M_{\Sigma, V}(G)/G$, and the quasi-Poisson structure on $M_{\Sigma, V}(G)$ reduces to the quasi-Poisson structure on $M_{\Sigma', V'}(G)$.
- (2) Let $x, y \in V_+$ and let Σ' be obtained from Σ by a “corner connected sum”:



Let V' be the image of V in Σ' , i.e. with x and y identified (denoted z on the picture). In this case $M_{\Sigma', V'}(G) \cong M_{\Sigma, V}(G)$ (the isomorphism is induced by the gluing map $\Sigma \rightarrow \Sigma'$) and the quasi-Poisson structure on $M_{\Sigma', V'}(G)$ is obtained from the quasi-Poisson structure on $M_{\Sigma, V}(G)$ by fusion of the \mathfrak{g} 's acting at x and y . Indeed, the additional term $(\rho_2 \otimes \rho_1)(t)$

in (5) corresponds to the new intersections close to the identified x and y . The same works for $x, y \in V_-$ if we reverse the order of x and y in the corner connected sum.

Using these operation we can get to arbitrary $(\Sigma, V_+ \sqcup V_-)$. \square

5.3. Centers of moduli spaces. The set $V \subset \partial\Sigma$ splits $\partial\Sigma$ to finitely many *boundary arcs*. There may be boundary circles without any marked points; we shall call them *uncut circles*. We shall say that a boundary arc from $x \in V$ to $y \in V$ (the arc is oriented by the orientation induced from Σ) is *left* if $x \in V_-$ and $y \in V_+$ (i.e. if the corresponding $\hat{x}, \hat{y} \in \hat{V}$ are “outside” of the arc) and *right* if $x \in V_+$ and $y \in V_-$ (i.e if \hat{x}, \hat{y} are “inside”). Notice that the number of the left arcs is equal to the number of the right arcs; there may be arcs which are neither left nor right.

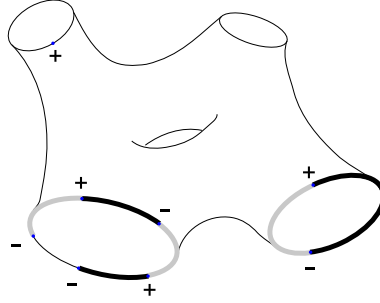


FIGURE 1. Left boundary arcs are **light** and right are **dark**; the remaining parts of the boundary are thin. Points in V_+ are marked by $+$ and in V_- by $-$.

Let L be the set of the left arcs. The holonomies along the left arcs give us a map

$$\mu_L : M_{\Sigma, V}(G) \rightarrow G^L, \quad f \mapsto (f(a))_{a \in L}$$

and likewise, if R is the set of the right arcs, we get a map

$$\mu_R : M_{\Sigma, V}(G) \rightarrow G^R, \quad f \mapsto (f(a))_{a \in R}.$$

Theorem 7. *The map μ_L is left-central and the map μ_R is right-central.*

Proof. Let $[c] \in H_1(\Sigma, V; \mathfrak{g}_{P,A}^*)$ be such that c is supported on the left arcs. Then for any $[c'] \in H_1(\Sigma, V; \mathfrak{g}_{P,A}^*)$ we have $\sigma_{V_+, V_-}([c], [c']) = 0$, as the supports of c and of the appropriately deformed c' don't intersect. This proves that μ_L is left-central. The proof for μ_R is similar. \square

When $t \in (S^2 \mathfrak{g})^{\mathfrak{g}}$ is non-degenerate, we can describe the left and the right leaves of $M_{\Sigma, V}(G)$ explicitly.

Theorem 8. *If t is non-degenerate then the left (right) leaves of $M_{\Sigma, V}(G)$ are obtained by fixing the holonomies along the left (right) boundary arcs (i.e. the value of $\mu_{L(R)}$) and the conjugacy classes of the holonomies along the uncut boundary circles. The big leaves are obtained by fixing only the conjugacy classes along the uncut circles.*

Proof. By Poincaré duality the intersection pairing

$$H_1(\Sigma, \partial\Sigma - \hat{V}; \mathfrak{g}_{P,A}^*) \times H_1(\Sigma, \hat{V}; \mathfrak{g}_{P,A}^*) \rightarrow \mathbb{R}$$

is non-degenerate. The left kernel of the pairing σ_{V_+, V_-} , i.e. the annihilator of $T^L M_{\Sigma, V}(G)$, is thus the kernel of the map

$$T^* M_{\Sigma, V}(G) = H_1(\Sigma, V; \mathfrak{g}_{P, A}^*) \rightarrow H_1(\Sigma, \partial\Sigma - \hat{V}; \mathfrak{g}_{P, A}^*),$$

i.e. (using the long exact sequence for the triple $V \subset \partial\Sigma - \hat{V} \subset \Sigma$) the image of

$$(10) \quad H_1(\partial\Sigma - \hat{V}, V; \mathfrak{g}_{P, A}^*) \rightarrow H_1(\Sigma, V; \mathfrak{g}_{P, A}^*).$$

For any connected component $K \subset \Sigma - \hat{V}$ we have $H_1(K, K \cap V; \mathfrak{g}_{P, A}^*) = 0$ unless K is a left arc (when $K \cap V$ contains two points) or if K is an uncut circle. The image of $H_1(\partial\Sigma - \hat{V}, V; \mathfrak{g}_{P, A}^*) = \bigoplus_K H_1(K, K \cap V; \mathfrak{g}_{P, A}^*)$ under the map (10) is the annihilator of the tangent space of the submanifold given by fixing the left holonomies and the uncut conjugacy classes. The proof for right leaves is similar, and the big leaves can be found using $T^{big} = T^L + \text{Im } \rho$. \square

5.4. The split-symplectic case. Theorem 8 enables us to single out the case when $M_{\Sigma, V}(G)$ is split-symplectic.

Theorem 9. *If t is non-degenerate, V meets every component of $\partial\Sigma$, and if the points in V_+ and V_- alternate along $\partial\Sigma$ (i.e. if every boundary arc is either left or right) then*

$$\begin{array}{ccc} & M_{\Sigma, V_+ \sqcup V_-}(G) & \\ \mu_L \swarrow & & \searrow \mu_R \\ G^L & & G^R \end{array}$$

is a split-symplectic central pair. The same is true if we allow uncut circles, and replace $M_{\Sigma, V_+ \sqcup V_-}(G)$ with any of its big leaves (i.e. if we fix the conjugacy class for every uncut circle).

Proof. We need to verify that the stabilizers of points in G^L and G^R are Lagrangian. We already know that they are coisotropic, and they are isomorphic to \mathfrak{g}^{V_+} , so they must be Lagrangian for dimension reasons. \square

In the split-symplectic case we can somewhat simplify the formula for the quasi-Poisson structure, and also express it in terms of differential forms. Let $\mathcal{L} \subset \partial\Sigma$ be the union of the left arcs and let $\mathcal{R} \subset \partial\Sigma$ the union of the right arcs, so that $\mathcal{L} \cup \mathcal{R} = \partial\Sigma$ and $\mathcal{L} \cap \mathcal{R} = V$ (for simplicity we treat the case with no uncut circles). By Poincaré duality the intersection pairing

$$H_1(\Sigma, \mathcal{L}; \mathfrak{g}_{P, A}^*) \times H_1(\Sigma, \mathcal{R}; \mathfrak{g}_{P, A}^*) \rightarrow \mathbb{R}$$

is non-degenerate. If we compose it with the maps (coming from the inclusions $V \subset \mathcal{L}, V \subset \mathcal{R}$)

$$H_1(\Sigma, V; \mathfrak{g}_{P, A}^*) \rightarrow H_1(\Sigma, \mathcal{L}; \mathfrak{g}_{P, A}^*), \quad H_1(\Sigma, V; \mathfrak{g}_{P, A}^*) \rightarrow H_1(\Sigma, \mathcal{R}; \mathfrak{g}_{P, A}^*)$$

we get the pairing σ on $T^*M = H_1(\Sigma, V; \mathfrak{g}_{P, A}^*)$ where $M = M_{\Sigma, V}(G)$.

In terms of cohomology, we have

$$T^L M = H^1(\Sigma, \mathcal{L}; \mathfrak{g}_{P, A}), \quad T^R M = H^1(\Sigma, \mathcal{R}; \mathfrak{g}_{P, A})$$

included in

$$TM = H^1(\Sigma, V; \mathfrak{g}_{P, A}).$$

The intersection pairing (non-degenerate by Poincaré duality)

$$H^1(\Sigma, \mathcal{L}; \mathfrak{g}_{P, A}) \times H^1(\Sigma, \mathcal{R}; \mathfrak{g}_{P, A}) \rightarrow \mathbb{R}$$

is then the pairing

$$\sigma^{-1} : T^L M \times T^R M \rightarrow \mathbb{R}$$

inverse to σ . We can express it in terms of differential forms: if we represent a cohomology class from $H^1(\Sigma, \mathcal{L}; \mathfrak{g}_{P,A})$ by a 1-form $\alpha \in \Omega^1(\Sigma, \mathfrak{g}_{P,A})$, $\alpha|_{\mathcal{L}} = 0$, and similarly a class from $H^1(\Sigma, \mathcal{R}; \mathfrak{g}_{P,A})$ by $\beta \in \Omega^1(\Sigma, \mathfrak{g}_{P,A})$, $\beta|_{\mathcal{R}} = 0$, then

$$\sigma^{-1}([\alpha], [\beta]) = \int_{\Sigma} \langle \alpha \wedge \beta \rangle.$$

5.5. Examples of reduced spaces. Let us now discuss some simple examples of reductions of $M_{\Sigma,V}(G)$'s. If $x \in V$ then $M_{\Sigma,V-\{x\}}(G) = M_{\Sigma,V}(G)/G$, where G acts at x ; the quasi-Poisson structure from $M_{\Sigma,V}(G)$ descends to the quasi-Poisson structure on $M_{\Sigma,V-\{x\}}(G)$. This is a reduction in the sense of Proposition 4 (or of Proposition 2 if $V = \{x\}$).

As a slightly more complicated example, let $x \in V_+$ and $y \in V_-$. Let us consider the Lie algebra $\mathfrak{g} \oplus \bar{\mathfrak{g}}$ acting at x and y and its diagonal subalgebra $\mathfrak{g}_{diag} \subset \mathfrak{g} \oplus \bar{\mathfrak{g}}$, which is coisotropic. In this case $M_{\Sigma,V}(G)/G_{diag} = M_{\Sigma',V-\{x,y\}}(G)$, where Σ' is obtained by joining x and y :



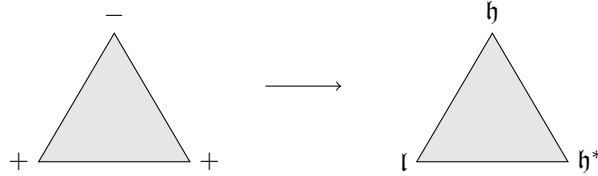
As the simplest example of central reduction, let us suppose that Σ has a single boundary circle and that $V_+ = \{x\}$, $V_- = \{y\}$. The circle $\partial\Sigma$ is cut by x and y to a left and a right arc. We thus have a central pair

$$\begin{array}{ccc} & M_{\Sigma,\{x,y\}}(G) & \\ \mu_L \swarrow & & \searrow \mu_R \\ G & & G \end{array}.$$

Reduction by $\mathfrak{g}_{diag} \subset \mathfrak{g} \oplus \bar{\mathfrak{g}}$ will replace Σ by Σ' with two boundary circles. Choice of a \mathfrak{g}_{diag} orbit in each G in the central pair corresponds to a choice of a conjugacy class for each of the two circles.

As the final example, let $\mathfrak{h}, \mathfrak{h}^*, \mathfrak{l} \subset \mathfrak{g}$ be Lagrangian Lie subalgebras. Let us suppose that $\mathfrak{h} \cap \mathfrak{h}^* = 0$, so that $\mathfrak{h}, \mathfrak{h}^* \subset \mathfrak{g}$ is a Manin triple, and thus \mathfrak{h} a Lie bialgebra. By a theorem of Drinfeld [D2], \mathfrak{l} defines a Poisson structure on $H/H \cap L$ which makes it to a Poisson H -space (for simplicity we shall work with local groups so that we don't have to spell out closedness conditions), and in this way we get a classification of Poisson homogeneous H -spaces.

To obtain this Poisson homogeneous space by reduction, let (Σ, V) be a triangle and let us reduce its moduli space by $\mathfrak{c} := \mathfrak{l} \oplus \mathfrak{h}^* \oplus \mathfrak{h} \subset \mathfrak{g} \oplus \mathfrak{g} \oplus \bar{\mathfrak{g}}$, as on the figure:



If we constrain the holonomy along the left edge (which defines a right-central map to G) to be in the \mathfrak{c} -orbit passing through 1, we get the Poisson homogeneous space $H/H \cap L$.

There are many other examples connected with the world of Poisson-Lie groups which can be obtained by reduction of moduli spaces. Some of them were studied

in [LS1, LS4] using moment map reduction. As we shall see in the next section, such a reduction is a special case of central reduction.

6. MOMENT MAPS VIA CENTERS; FUSION OF D/H -MOMENT MAPS

In this section we shall see that central maps and central reduction contains, as a special case, the theory of (quasi-)Hamiltonian spaces and their reduction. In particular, if $X \subset M$ is a left leaf of a split-symplectic \mathfrak{d} -manifold and if $\mathfrak{h} \subset \mathfrak{d}$ is the stabilizer of X then X carries a natural $(\mathfrak{h}, \mathfrak{d})$ -quasi-symplectic structure and the map from X to the (local) space of the right leaves of M is a (local) moment map.

This point of view also allows us to define the fusion of D/H -valued moment maps, which was so far missing.

Throughout this section \mathfrak{d} denotes a Lie algebra with an invariant non-degenerate symmetric pairing.

Let $\mathfrak{h} \subset \mathfrak{d}$ be a Manin pair, i.e. \mathfrak{d} is a Lie algebra with an invariant non-degenerate symmetric pairing, and \mathfrak{h} is its Lagrangian subalgebra. Let $\mathfrak{h}^* \subset \mathfrak{d}$ be a Lagrangian complement of \mathfrak{h} . Equivalently, \mathfrak{h} is a quasi-Lie bialgebra, with $\delta_{\mathfrak{h}} : \mathfrak{h} \rightarrow \mathfrak{h} \wedge \mathfrak{h}$ and $\phi_{\mathfrak{h}} \in \bigwedge^3 \mathfrak{h}$ given by the \mathfrak{h}^* and \mathfrak{h} components of

$$\mathfrak{h}^* \otimes \mathfrak{h}^* \subset \mathfrak{d} \otimes \mathfrak{d} \xrightarrow{[\cdot, \cdot]} \mathfrak{d} \cong \mathfrak{h} \oplus \mathfrak{h}^*.$$

Definition 10 ([AK]). *An $(\mathfrak{h}, \mathfrak{d}; \mathfrak{h}^*)$ -quasi-Poisson manifold is an \mathfrak{h} -manifold M with a bivector field π such that*

$$\begin{aligned} [\pi, \pi]/2 &= \rho_M^{\otimes 3}(\phi_{\mathfrak{h}}) \\ [\rho_M(u), \pi] &= -\rho_M^{\otimes 2}(\delta_{\mathfrak{h}}(u)) \quad (\forall u \in \mathfrak{h}). \end{aligned}$$

If $\mathfrak{h}^* \subset \mathfrak{d}$ is a Lie subalgebra (i.e. if $\mathfrak{h}, \mathfrak{h}^* \subset \mathfrak{d}$ is a Manin triple and \mathfrak{h} a Lie bialgebra) then $\phi_{\mathfrak{h}} = 0$ and π is thus a Poisson structure.

For any $(\mathfrak{h}, \mathfrak{d}; \mathfrak{h}^*)$ -quasi-Poisson manifold M the distribution given by the image of

$$a : \mathfrak{h} \oplus T^*M \rightarrow TM, \quad (u, \alpha) \mapsto \rho(u) + \pi(\alpha, \cdot)$$

is integrable, as a is the anchor map of a Lie algebroid structure (see [BCS]). Its integral leaves are the minimal $(\mathfrak{h}, \mathfrak{d}; \mathfrak{h}^*)$ -quasi-Poisson submanifolds of M , and are called the *quasi-symplectic leaves* of M . M is *quasi-symplectic* if it contains just one quasi-symplectic leaf.

Notice that Definition 1 is a special case of Definition 10. If \mathfrak{g} is a Lie algebra with an invariant element $t \in S^2 \mathfrak{g}$, and if we suppose for simplicity that t is non-degenerate, then $\mathfrak{g}_{\text{diag}} \subset \mathfrak{g} \oplus \mathfrak{g}$ is a Manin pair. A \mathfrak{g} -quasi-Poisson structure is the same as a $(\mathfrak{g}_{\text{diag}}, \mathfrak{g} \oplus \mathfrak{g}; \mathfrak{g}_{\text{antidiag}})$ -quasi-Poisson structure.

Definition 10 needs to be complemented by an explanation of what happens if we change the complement $\mathfrak{h}^* \subset \mathfrak{d}$, as \mathfrak{h}^* is understood as auxiliary data. Lagrangian complements are in 1-1 correspondence with elements $\tau \in \bigwedge^2 \mathfrak{h}$ via

$$\tau \mapsto \mathfrak{h}_{\tau}^* := \{(\tau(\cdot, \alpha), \alpha); \alpha \in \mathfrak{h}^*\} \subset \mathfrak{h} \oplus \mathfrak{h}^* \cong \mathfrak{d}$$

(so that $\mathfrak{h}_0^* = \mathfrak{h}^*$). If we replace \mathfrak{h}^* by \mathfrak{h}_{τ}^* then π has to be replaced by $\pi - \rho_M^{\otimes 2}(\tau)$. The element τ is called a *twist*. See [AK] for details.

Let us observe that any \mathfrak{d} -quasi-Poisson manifold M carries also a $(\mathfrak{h}, \mathfrak{d}; \mathfrak{h}^*)$ -quasi-Poisson structure. Indeed, let e_i be a basis of \mathfrak{h} and e^i the dual basis of the complement $\mathfrak{h}^* \subset \mathfrak{d}$. The twist

$$\tau_{\mathfrak{h}, \mathfrak{h}^*} = \frac{1}{2} \sum_i e_i \wedge e^i \in \bigwedge^2 \mathfrak{d}$$

corresponds to the new complement $\mathfrak{h} \oplus \mathfrak{h}^* \subset \mathfrak{d} \oplus \bar{\mathfrak{d}}$ of $\mathfrak{d}_{\text{diag}} \subset \mathfrak{d} \oplus \bar{\mathfrak{d}}$. With this new complement \mathfrak{h} is a quasi-Lie sub-bialgebra¹ of \mathfrak{d} , hence M with the action of \mathfrak{h} and with

$$(11) \quad \pi' = \pi - \rho^{\otimes 2}(\tau_{\mathfrak{h}, \mathfrak{h}^*})$$

is $(\mathfrak{h}, \mathfrak{d}; \mathfrak{h}^*)$ -quasi-Poisson.

Theorem 10. *Let (M, ρ, π) be a split-symplectic \mathfrak{d} -manifold and let $X \subset M$ be a left leaf. Let $\mathfrak{h} \subset \mathfrak{d}$ be the stabilizer of X , and let $\mathfrak{h}^* \subset \mathfrak{d}$ be a Lagrangian complement of \mathfrak{h} . Then the bivector field (11) is tangent to X and $(X, \rho|_X, \pi'|_X)$ is a $(\mathfrak{h}, \mathfrak{d}; \mathfrak{h}^*)$ -quasi-symplectic manifold.*

Proof. As $(M, \rho|_{\mathfrak{h}}, \pi')$ is $(\mathfrak{h}, \mathfrak{d}; \mathfrak{h}^*)$ -quasi-Poisson, it remains to show that π' is tangent to X and that the image of

$$a : \mathfrak{h} \oplus T^*X \rightarrow TX, (u, \alpha) \mapsto \rho(u) + \pi'(\alpha, \cdot)$$

is TX . As

$$t/2 + \tau_{\mathfrak{h}, \mathfrak{h}^*} = \sum_i e_i \otimes e^i,$$

we get

$$\pi' = \sigma - \sum_i \rho(e_i) \otimes \rho(e^i),$$

which implies, for every $x \in X$ and $\alpha \in T_x^*M$,

$$\pi'(\cdot, \alpha) = \sigma(\cdot, \alpha) - \sum_i \alpha(\rho(e^i))\rho(e_i) \in T_x^R M + \rho_x(\mathfrak{h}) = T_x^R M = T_x X.$$

This also shows that the image of a is equal to the image of $\alpha \mapsto \sigma(\cdot, \alpha)$, i.e. to $T_x X$. \square

Let us now recall the definition of moment maps.

Definition 11 ([AK, LS4]). *If (M, ρ, π) is a $(\mathfrak{h}, \mathfrak{d}; \mathfrak{h}^*)$ -quasi-Poisson manifold and if N is a \mathfrak{d} -manifold with coisotropic stabilizers then a map $\mu : M \rightarrow N$ is a N -valued moment map if it is \mathfrak{h} -equivariant and if*

$$(1 \otimes \mu_*)(\pi) = -(\rho_M \otimes 1)(Z_N),$$

where $Z_N \in \Gamma(\mathfrak{h} \otimes TN)$ is given by

$$\langle \alpha, Z_N \rangle = \rho_N(\alpha) \quad \forall \alpha \in \mathfrak{h}^* \subset \mathfrak{d}$$

and ρ_N is the action of \mathfrak{d} on N .

The most important cases are $N = D/H$, where $H \subset D$ is the connected group integrating \mathfrak{h} (provided $H \subset D$ is closed, or working with local groups), and more generally $N = D/H'$, where $\mathfrak{h}' \subset \mathfrak{d}$ is another Lagrangian Lie subalgebra. When $\delta_{\mathfrak{h}} = 0$ and $\phi_{\mathfrak{h}} = 0$, i.e. when M has a \mathfrak{h} -invariant Poisson structure, we have $D \cong H \ltimes \mathfrak{h}^*$ and $D/H \cong \mathfrak{h}^*$; in this case a D/H -valued moment map is simply a moment map in the usual sense.

Remark. Suppose that $t \in (S^2 \mathfrak{g})^{\mathfrak{g}}$ is nondegenerate, so that $D = G \times G$. We can identify D/G (where $G \subset D$ is the diagonal) with G . If M is a \mathfrak{g} -quasi-Poisson manifold, a moment map $\mu : M \rightarrow D/G = G$ is called in [AMM, AKM] a group-valued moment map.

¹ If \mathfrak{a} is a quasi-Lie bialgebra then a Lie subalgebra $\mathfrak{b} \subset \mathfrak{a}$ is a quasi-Lie sub-bialgebra if $\delta_{\mathfrak{a}}$ restricted to \mathfrak{b} has values in $\bigwedge^2 \mathfrak{b}$ and $\phi_{\mathfrak{a}} \in \bigwedge^3 \mathfrak{b} \subset \bigwedge^3 \mathfrak{a}$. This makes \mathfrak{b} with $\delta_{\mathfrak{b}} := \delta_{\mathfrak{a}}|_{\mathfrak{b}}$ and $\phi_{\mathfrak{b}} := \phi_{\mathfrak{a}}$ to a quasi-Lie bialgebra.

In this case left (and right) leaves of M coincide with the big leaves [LS3, Theorem 3]. In particular, if M is quasi-symplectic then σ is non-degenerate. One can show that

$$(12) \quad \sigma^{-1} = \omega + \frac{1}{2}\mu^*s \in \Gamma((T^*)^{\otimes 2}M)$$

where $\omega \in \Omega^2(M)$ is the skew-symmetric part of σ^{-1} and s is the bi-invariant (pseudo-)Riemann metric on G given by t . The 2-form ω , together with the action of \mathfrak{g} and the map μ , makes M to a quasi-Hamiltonian space in the sense of [AMM]. The relation between π and ω given by (12) is somewhat cleaner than the (equivalent) relation given in [AKM], as it clarifies the role of non-degeneracy conditions, which are simply the invertibility of both sides of Equation (12).

Theorem 10 can now be complemented as follows.

Theorem 11. *If, in the context of Theorem 10, $\mu_R : M \rightarrow N$ is a right-central map then $\mu_R|_X : X \rightarrow N$ is a moment map.*

Proof. Right-centrality of μ_R means $(1 \otimes (\mu_R)_*)(\sigma) = 0$, hence

$$\begin{aligned} (1 \otimes (\mu_R)_*)(\pi') &= -(1 \otimes (\mu_R)_*)\left(\sum_i \rho_M(e_i) \otimes \rho_M(e^i)\right) = \\ &= -\sum_i \rho_M(e_i) \otimes \rho_N(e^i) = -(1 \otimes \rho_M)(Z_N). \end{aligned}$$

This shows that $\mu_R : M \rightarrow N$ is a moment map for the $(\mathfrak{h}, \mathfrak{d}; \mathfrak{h}^*)$ -quasi-Poisson structure $\pi' = \pi - \rho^{\otimes 2}(\tau_{\mathfrak{h}, \mathfrak{h}^*})$ on M , and thus $\mu_R|_X$ is a moment map, too. \square

Example 4. Let us return to moduli spaces of flat connections. In the case when $V = V_+$ the map $M_{\Sigma, V_+ \sqcup \emptyset}(G) \rightarrow G^n$ given by the holonomies along the boundary arcs (where $n = |V_+|$ is the number of boundary arcs) is a moment map. This observation made in [AMM, AKM] was the reason why the authors of *op. cit.* started developing the theory of group-valued moment maps and quasi-Poisson geometry. Let us explain these moment maps using central pairs. For simplicity we suppose (as in *op. cit.*) that t is non-degenerate and that V_+ intersects every component of $\partial\Sigma$ (which implies that $M_{\Sigma, V_+ \sqcup \emptyset}(G)$ is quasi-symplectic).

Let us choose a point on each of the n boundary arcs and let V_- be the set of these new points. Then $M = M_{\Sigma, V_+ \sqcup V_-}(G)$ is as in Theorem 9, i.e.

$$\begin{array}{ccc} & M & \\ \mu_L \swarrow & & \searrow \mu_R \\ G^L & & G^R \end{array}$$

is a split-symplectic central pair. This implies that $X := \mu_L^{-1}(1) \subset M$ (as any fibre of μ_L) is a left leaf. Since $X \subset M$ is given by the condition that the holonomies along the left boundary arcs are equal to 1, we can identify X (by contracting the left arcs) with $M_{\Sigma, V_+ \sqcup \emptyset}(G)$. The quasi-Poisson structure on X given by Theorem 10 is equal to the original quasi-Poisson structure on $M_{\Sigma, V_+ \sqcup \emptyset}(G)$.

Theorem 11 now says that $\mu_R|_X : X \rightarrow G^R = G^n$ is a moment map, and it is the original moment map $M_{\Sigma, V_+ \sqcup \emptyset}(G) \rightarrow G^n$.

To make the link between centers and moment maps complete, we need to be more specific about the category we work in, namely choose one of these possibilities:

- (1) If D denotes a connected group integrating \mathfrak{d} and $H \subset D$ the connected subgroup integrating \mathfrak{h} , we need to suppose that H is closed in D . Moreover

we should consider only \mathfrak{d} - and \mathfrak{h} -actions which integrate to D - and H -actions.

- (2) Without imposing any restrictions, we can work with local groups D and H . In this case D/H denotes any manifold with a transitive \mathfrak{d} -action and with a chosen point $[1] \in D/H$ whose stabilizer is $\mathfrak{h} \subset \mathfrak{d}$. We then have to understand the results in the appropriate local form.

To stress that we now replace \mathfrak{h} - and \mathfrak{d} -actions with H - and D -actions, we shall use terminology “ D -quasi-Poisson manifolds”, “ $(H, \mathfrak{d}; \mathfrak{h}^*)$ -quasi-Poisson manifolds”, etc.

Theorem 12. *There is 1-1 correspondence between D -central pairs*

$$(13) \quad \begin{array}{ccc} & M & \\ \mu_L \swarrow & & \searrow \mu_R \\ D/H & & N \end{array}$$

and $(H, \mathfrak{d}; \mathfrak{h}^*)$ -quasi-Poisson manifolds X with a moment map

$$(14) \quad \mu : X \rightarrow N.$$

The correspondence is: If the central pair (13) is given then $X = \mu_L^{-1}([1])$ and $\mu = \mu_R|_X$. If X and μ are given, then M is obtained from X by inducing the H -action to a D -action, i.e.

$$(15) \quad M = (D \times X)/H$$

where H acts on $D \times X$ via $h \cdot (d, x) = (dh^{-1}, h \cdot x)$; the action of D on M is $d' \cdot [(d, x)] = [(d'd, x)]$. The central maps are

$$\mu_L([(d, x)]) = [d] \in D/H, \quad \mu_R([(d, x)]) = d \cdot \mu(x) \in N.$$

The link between the bivector field π on M and π' on $X = \mu_L^{-1}([1])$ is given by (11).

Proof. As the action of \mathfrak{d} on D/H is transitive, the map μ_L is a submersion, and thus $\mu_L^{-1}([1]) \subset M$ is a submanifold.

If we forget about bivector fields and understand (13) and (14) as diagrams of D - and H -equivariant maps respectively, then their equivalence is simply the universal property of the induction (15) from H -action to D -action.

The bivector fields can be treated as follows. If the central pair (13) is given then, as in the proof of Theorem 10, (11) makes M to a $(\mathfrak{h}, \mathfrak{d}; \mathfrak{h}^*)$ -quasi-Poisson manifold, and it is tangent to X , hence $X \subset M$ is a $(\mathfrak{h}, \mathfrak{d}; \mathfrak{h}^*)$ -quasi-Poisson manifold. As in the proof of Theorem 11, $\mu_R : M \rightarrow N$ is a moment map, and thus restricts to a moment map $X \rightarrow N$.

For the other direction, suppose that the bivector field π' on X is given. We define π first as a section of $(TM)|_X$ via

$$\pi = \pi' + \rho_M^{\otimes 2}(\tau_{\mathfrak{h}, \mathfrak{h}^*}).$$

The property

$$[\rho_X(u), \pi'] = -\rho_X^{\otimes 2}(\delta(u)) \quad (\forall u \in \mathfrak{h})$$

ensures that π is H -invariant. We can thus extend it (uniquely) to a D -invariant bivector field on M . The fact that π' is $(\mathfrak{h}, \mathfrak{d}; \mathfrak{h}^*)$ -quasi-Poisson then implies that $[\pi, \pi]/2 = \rho_M^{\otimes 3}(\phi_{\mathfrak{d}})$ at the points of X ; as π is D -invariant, this relation is satisfied everywhere on M , i.e. π is \mathfrak{d} -quasi-Poisson. Left and right centrality of μ_L and μ_R (first at the points of $X \subset M$ and then on entire M by D -invariance) then follows easily from $\sigma = \pi' + \sum_i \rho(e_i) \otimes \rho(e^i)$ and from the fact that μ is a moment map. \square

We can now explain why moment map reduction, in its most general form given in [LS4], is a special case of central reduction. Namely, if $C \subset D$ is a Lagrangian subgroup and $\mathcal{O}_N \subset N$ is a C -invariant submanifold, and if $\mu : X \rightarrow N$ is a moment map, then the reduction theorem of *op. cit.* makes $X_{red} := \mu^{-1}(\mathcal{O}_N)/C \cap H$ to a Poisson manifold; if X is quasi-symplectic, the action on N is transitive with Lagrangian stabilizers, and \mathcal{O}_N is a C -orbit, then X_{red} is symplectic. If we induce X to a central pair (13) then this reduction is simply the central reduction of M with $\mathcal{O}_N \subset N$ and $\mathcal{O}_{D/H} = C \cdot [1] \subset D/H$.

As another application, we can define fusion product of D/H -valued moment maps. If $\mu : X_1 \rightarrow D/H$ and $\nu : X_2 \rightarrow D/H$ are moment maps for $(H, \mathfrak{d}; \mathfrak{h}^*)$ -quasi-Poisson manifolds X_1 and X_2 , we induce them to central pairs

$$\begin{array}{ccc} M_1 & & M_2 \\ \mu_L \swarrow & & \searrow \mu_R \\ D/H & & D/H \end{array} \quad \begin{array}{ccc} M_2 & & M_2 \\ \nu_L \swarrow & & \searrow \nu_R \\ D/H & & D/H \end{array} .$$

By Theorem 5,

$$\begin{array}{ccc} M_1 \times_{D/H} M_2 & & \\ \mu_L \swarrow & & \searrow \nu_R \\ D/H & & D/H \end{array}$$

is a central pair, and we define, using our correspondence, the fusion of X_1 and X_2 as the $(H, \mathfrak{d}; \mathfrak{h}^*)$ -quasi-Poisson manifold

$$X_1 \otimes X_2 := \mu_L^{-1}([1]) \subset M_1 \times_{D/H} M_2.$$

In other words,

$$X_1 \otimes X_2 = X_1 \times_{D/H} (D \times X_2)/H,$$

where the fibre product is taken over the maps $\mu : X_1 \rightarrow N$ and $\nu_L : (D \times X_2)/H \rightarrow D/H$. The fact that $X_1 \otimes X_2$ is not $X_1 \times X_2$ is probably the reason why it was so far elusive.

Similarly, we can define the conjugate $\bar{\mu} : \bar{X} \rightarrow D/H$ of a D/H -valued moment map $\mu : X \rightarrow D/H$. Let

$$\begin{array}{ccc} M = (D \times X)/H & & \\ \mu_L \swarrow & & \searrow \mu_R \\ D/H & & D/H \end{array}$$

be the central pair corresponding to $\mu : X \rightarrow D/H$ by Theorem 12. Let $\bar{M} = M$ with π replaced by $-\pi$. \bar{M} is still a D -quasi-Poisson manifold, but $\mu_L, \mu_R : \bar{M} \rightarrow D/H$ are now right and left central respectively. As a result,

$$\bar{X} := \mu_R^{-1}([1]) \subset \bar{M}$$

is $(H, \mathfrak{d}; \mathfrak{h}^*)$ -quasi-Poisson and $\bar{\mu} := \mu_L|_{\bar{X}}$ is a moment map.

Example 5. D/H is a commutative \mathfrak{d} -quasi-Poisson manifold, and we have the central pair

$$\begin{array}{ccc} D/H \otimes D/H & & \\ p_1 \swarrow & & \searrow p_2 \\ D/H & & D/H \end{array}$$

where $p_{1,2}$ are the projections. Our correspondence makes $p_1^{-1}([1]) = D/H$ to a (H, \mathfrak{d}) -quasi-Poisson manifold with the moment map $\text{id} : D/H \rightarrow D/H$. To avoid confusion with the \mathfrak{d} -quasi-Poisson D/H , let us denote this (H, \mathfrak{d}) -quasi-Poisson manifold by $(D/H)_{(\mathfrak{h})}$. It was discovered in [AK]. The central pair corresponding the fusion product of $(D/H)_{(\mathfrak{h})}$ with itself is readily seen to be

$$\begin{array}{ccc} & D/H \otimes D/H \otimes D/H & \\ p_1 \swarrow & & \searrow p_3 \\ D/H & & D/H \end{array}$$

where $p_{1,3}$ are the projections to the first and the third factor respectively.

APPENDIX A. A NON-DEGENERACY LEMMA

All the vector spaces in this section are over a field K , $\text{char } K \neq 2$, and finite-dimensional.

Let U and U' be vector spaces, and let $\langle \cdot, \cdot \rangle : U \times U' \rightarrow K$ be a non-degenerate pairing. Let us introduce the following bilinear forms on $U \oplus U'$:

$$\begin{aligned} (u \oplus \alpha, v \oplus \beta) &= \langle u, \beta \rangle \\ (u \oplus \alpha, v \oplus \beta)_{\text{sym}} &= \langle u, \beta \rangle + \langle v, \alpha \rangle \\ (u \oplus \alpha, v \oplus \beta)_{\text{skew}} &= \langle u, \beta \rangle - \langle v, \alpha \rangle. \end{aligned}$$

A subspace $L \subset U \oplus U'$ is $(\cdot, \cdot)_{\text{sym}}$ -Lagrangian if $(x, y)_{\text{sym}} = 0$ for all $y \in L$ iff $x \in L$.

Proposition 5. *If $L \subset U \oplus U'$ is $(\cdot, \cdot)_{\text{sym}}$ -Lagrangian then*

$$\ker((\cdot, \cdot)_{\text{skew}}|_L) = (L \cap U) \oplus (L \cap U').$$

Proof. Notice that $(\cdot, \cdot)_{\text{skew}}|_L = 2(\cdot, \cdot)|_L$, as $(\cdot, \cdot)_{\text{sym}}|_L = 0$. This shows that both $(L \cap U)$ and $(L \cap U')$ are in the kernel, as U and U' are the right and left kernel of (\cdot, \cdot) .

If, on the other hand, $u \oplus \alpha \in \ker((\cdot, \cdot)_{\text{skew}}|_L) = \ker((\cdot, \cdot)|_L)$ then, for every $x \in U \oplus U'$, $(u \oplus \alpha, x)_{\text{sym}} = (u \oplus 0, x) = (u \oplus \alpha, x) = 0$. Since L is Lagrangian, this implies $u \oplus 0 \in L \cap U$ and thus also $0 \oplus \alpha \in L \cap U'$, hence $u \oplus \alpha \in (L \cap U) \oplus (L \cap U')$. \square

Proposition 6. *Let V be a vector space with a bilinear pairing $\sigma : V \times V \rightarrow K$, and W a vector space with a symmetric non-degenerate pairing $t : W \times W \rightarrow K$. Let $f : V \rightarrow W$ be a linear map such that*

$$\sigma(v, v) = \frac{1}{2}t(f(v), f(v))$$

for every $v \in V$. Let $V_L, V_R \subset V$ be the left and right kernels of σ . Suppose that $V_L \cap \ker f = 0$. If $C \subset W$ is t -Lagrangian then the kernel of $\sigma|_{f^{-1}(C)}$ (which is a skew-symmetric form) is

$$\ker \sigma|_{f^{-1}(C)} = V_L \cap f^{-1}(C) + V_R \cap f^{-1}(C).$$

Proof. Let us consider the vector space

$$X = V/V_L \oplus V/V_R \oplus W$$

and the injective map

$$F : V \rightarrow X, \quad v \mapsto ([v], [v], f(v)).$$

On X we have the non-degenerate symmetric pairing $(\cdot, \cdot)_{\text{sym}} \oplus (-t)$, where the pairing $(\cdot, \cdot)_{\text{sym}}$ on $V/V_L \oplus V/V_R$ comes from the pairing $\langle [v], [v'] \rangle = \sigma(v, v')$, $[v] \in$

V/V_L , $[v'] \in V/V_R$. The image of V is isotropic, and for dimension reasons it is Lagrangian.

Since the composition of Lagrangian relations is Lagrangian, the image of the map

$$F' : f^{-1}(C) \rightarrow V/V_L \oplus V/V_R, v \mapsto ([v], [v])$$

is Lagrangian, being the composition of $F(V)$ and C . The kernel of F' is

$$(V_L \cap f^{-1}(C)) \cap (V_R \cap f^{-1}(C)).$$

Finally, since

$$\sigma|_{f^{-1}(C)}(x, y) = (F'(x), F'(y))_{skew},$$

the result follows from Proposition 5, applied to the Lagrangian subspace $F'(f^{-1}(C))$ of $V/V_L \oplus V/V_R$.

□

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SECTION OF MATHEMATICS, UNIVERSITY OF GENEVA, SWITZERLAND
E-mail address: pavel.severa@gmail.com